Math 279 Lecture 21 Notes

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1 Two Examples of Regularity Structures

We discuss two models for our theory before treating our ill-posed PDEs.

1.1 Finite Taylor Polynomials

Example 1.1. Let $A = \mathbb{N}$, $T = \mathbb{R}[X_1, \dots, X_d]$, with $T_r = \langle X^k : |k| = r \rangle$, and $\| \cdot \|_r$ the standard Euclidean norm. Recall that $\Gamma_{x,y} = \Gamma_{x-y}$, with $\Gamma_h(P(X)) = P(X + h\mathbf{1})$. For our model, $(\Pi_a(P(X)))(x) = P(x - a)$. This gives the model $M = (\Pi, \Gamma)$. We now specify $\mathcal{C}^\gamma_M = \{f : \mathbb{R}^d \to \bigoplus_{r<\gamma} T_r \mid \|f(x) - \Gamma_{x,y} f(y)\|_r \lesssim |x - y|^{\gamma - r}\}$.

We claim that for any $\gamma > 0$, $\mathcal{C}^\gamma_M$ is isomorphic to $\mathcal{C}^\gamma(\mathbb{R}^d)$. Let us assume that $\gamma = n + \gamma_0$ with $n \in \mathbb{N}$ and $\gamma_0 \in (0, 1)$. Then $f \in \mathcal{C}^\gamma_M$ means that $f(x)$ is a polynomial of degree at most n i.e. $f(x) = \sum_{k:|k|\leq n} a_k(x) X^k$, with (setting $h = x - y$ so that $x = y + h$)

$$\left\| \sum_{k:|k|\leq n} a_k(y + h) X^k - \sum_{k:|k|\leq n} a_k(y)(X + h\mathbf{1})^k \right\|_r \lesssim |h|^{\gamma - r}.$$

For example, if $r = n$,

$$\sum_{k:|k|\leq n} |a_k(y + h) - a_k(y)| \lesssim |h|^{\gamma_0},$$

which means that when $|k| = n$, $a_k(y)$ is γ_0-Hölder. More generally,

$$\sum_{|\ell|=r} \left| a_\ell(y + h) - \sum_{\substack{k:k\geq \ell \\ |k|\leq n}} \binom{k}{\ell} a_k(y) h^{k-\ell} \right| \lesssim |h|^{\gamma - r}.$$

To ease the notation, assume $d = 1$ and $r = n - 1$. Then we get

$$|a_{n-1}(y + h) - a_{n-1}(y) - n a_n(y) h| \lesssim |h|^{\gamma_0 + 1}$$

Divide by h and send $h \to 0$ to arrive at: a_{n-1} is differentiable, and $\frac{d}{dy}a_{n-1} = na_n$. Inductively, we can show that

$$a_k(y) = \frac{1}{k!}\partial^k a_0(y).$$

In summary,

$$f(x) = \sum_{|k|\le n} a_k(x)X^k \in \mathcal{C}_M^{\gamma_0+h} \iff a_0 \in \mathcal{C}^\gamma(\mathbb{R}^d), f(x) = \text{Taylor expansion of } \deg n \text{ of } a_0.$$

Remark 1.1 (Whitney expansion)**.** Imagine that a closed set $K \subseteq \mathbb{R}^d$ is given and we assign a polynomial $f(x)$ as above to each $x \in K$. If for $x \in K$ the bound

$$\left\| \sum_{k:|k|\le n} a_k(y+h)X^k - \sum_{k:|k|\le n} a_k(y)(X+h\mathbf{1})^k \right\|_r \lesssim |h|^{\gamma-r}.$$

holds, then $f(x)$ can serve as a candidate for the Taylor expansion of a suitable function $a_0 : \mathbb{R}^d \to \mathbb{R}$ such that for $x \in K$, $f(x)$ is indeed its Taylor expansion.

1.2 The Gubinelli derivative

Example 1.2. Pick $\alpha \in (1/3, 1/2)$, and choose $A = \{\alpha-1, 2\alpha-1, 0, \alpha\}$. Note that $r = 1$, i.e. the integer -1 is the best lower bound for A. We define

$$T_0 = \langle \mathbf{1} \rangle, \qquad T_\alpha = \langle X_1, X_2, \ldots, X_\ell \rangle,$$
$$T_{\alpha-1} = \langle \dot{X}_1, \dot{X}_2, \ldots, \dot{X}_\ell \rangle, \qquad T_{2\alpha-1} = \langle \dot{\mathbb{X}}^{i,j} : 1 \le i,j \le \ell \rangle,$$

so $T = \bigoplus_{\beta\in A} T_\beta$ has $\dim T = (\ell+1)^2$. Here, these are all just formal symbols, but we have written the notation to be suggestive. Next, $G = \{\Gamma_h : h \in \mathbb{R}^\ell\}$ with

$$\Gamma_h \mathbf{1} = \mathbf{1}, \qquad \Gamma_h X = X + h\mathbf{1},$$
$$\Gamma_h \dot{X} = \dot{X}, \qquad \Gamma_h \dot{\mathbb{X}} = \dot{\mathbb{X}} + h \otimes \dot{X}.$$

Next, we define a model. Given a rough path $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}_{\alpha,2\alpha}$, i.e. $x : [0,T] \to \mathbb{R}^\ell$, $\mathbb{X}(s,t) \in \mathbb{R}^{\ell\times\ell}$, and Chen's relation. We build a model as follows:

$$(\Pi_s \mathbf{1})(t) = 1, \qquad (\Pi_s X_i)(t) = x_i(s,t) = x_i(t) - x_i(s),$$
$$(\Pi_s(\dot{X}_i))(\underbrace{\psi}_{\in\mathcal{D}}) = (\dot{x}_i)(\psi) = -\int \dot{\psi}(t)x_i(t)\,dt,$$
$$(\Pi_s \dot{\mathbb{X}}^{i,j})(\psi) = (\mathbb{X}_t^{i,j}(s,\cdot))(\psi) = -\int \dot{\psi}(t)\mathbb{X}^{i,j}(s,t)\,dt.$$

Next, we have

$$\Gamma_{s,s'} = \Gamma_{x(s',s)}.$$

We need to verify a number of things:

- $\langle \Pi_s, \varphi_s^\delta \rangle = \int ((x(t) - x(s))\varphi(\frac{t-s}{s})\frac{1}{\delta} \lesssim \delta^\alpha$, which follows from $x \in \mathcal{C}^\alpha$.
- Similarly,
$$(\Pi_s(\dot{\mathbb{X}})(\varphi_s^\delta) = -\int \frac{d}{dt}\varphi_s(t)\mathbb{X}(s,t)\,dt = -\frac{1}{\delta}\int \dot{\varphi}(\theta)\mathbb{X}(s, s+\delta\theta)\,d\theta.$$
Hence,
$$|(\Pi_s\dot{\mathbb{X}})(\varphi_s^\delta)| \lesssim [\mathbb{X}]_{2\alpha}\delta^{2\alpha-1}\|\varphi\|_{C^1}.$$
- Next, we need to check $\Pi_{s'} = \Pi_s\Gamma_{s,s'}$. Indeed,
$$(\Pi_{s'}\dot{\mathbb{X}})(\psi) = -\int \dot{\psi}(t)\mathbb{X}(s',t)\,dt,$$
$$(\Pi_s\Gamma_{s,s'}\dot{\mathbb{X}})(\psi) = -\int \dot{\psi}(t)(\mathbb{X}(s,t) + x(s',s)\otimes x(t))\,dt$$
Since ψ is of 0 average,
$$= -\int \dot{\psi}(t)(\mathbb{X}(s,t) + x(s',s)\otimes x(s,t))\,dt$$
$$= -\int \dot{\psi}(t)(\mathbb{X}(s,t) + \mathbb{X}(s',s) + x(s',s)\otimes x(s,t))\,dt$$
By Chen's relation,
$$= \int \dot{\psi}(t)\mathbb{X}(s',t)\,dt,$$
as desired.

Next, we examine $\mathcal{C}_M^{2\alpha}$. Assume $Y \in \mathcal{C}_M^{2\alpha}$ is of the form
$$Y(t) = y(t)\mathbf{1} + \widehat{y}(t)\cdot X.$$
We claim that $T \in \mathcal{C}_M^{2\alpha}$ if and only if $\mathbf{y} = (y, \widehat{y}) \in \mathscr{G}_{\alpha,2\alpha}(\mathbf{x})$ (i.e. $\widehat{y}$ is a Gubinelli derivative of y). Indeed,
$$\|Y(t) - \Gamma_{t,t'}Y(t')\|_r \lesssim |t-t'|^{2\alpha-r}.$$
This is
$$\|y(t)\mathbf{1} + \widehat{y}(t)\cdot X - (y(t')\mathbf{1} + \widehat{y}(t')\cdot(X + x(t',t)\nVdash))\| \lesssim |t-t'|^{2\alpha-1}.$$
Choose $r = \alpha$. Then
$$|\widehat{y}(t) - \widehat{y}(t')| \lesssim |t-t'|^\alpha,$$
i.e. $\widehat{y} \in \mathcal{C}^\alpha$. Next, choose $r = 0$. We get
$$|y(t) - y(t') - \widehat{y}(t')x(t',t)| \lesssim |t-t'|^{2\alpha}.$$

Imagine that we want to make sense of $\mathbf{y}\cdot d\mathbf{x}$. This should really be the realization of $Y\cdot\dot{X}$. We will give a candidate for the abstract multiplication and recover our previous results using the reconstruction theorem.